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Deterministic One-Dimensional Cellular Automata

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A formal treatment of some of the properties of deterministic, rule 150, elementary one-dimensional cellular automata (CA) with null boundary conditions is presented. The general form of the characteristic polynomial of the CA global rule transition matrix is obtained. Mathematical relationships between the CA register lengths and the order of the corresponding group or semigroup structures are derived.

KEY WORDS: Automata theory; cellular automata; group theory; mesh-connected computers; VLSI; parallel processing; discrete dynamical systems.

1. INTRODUCTION

The various aspects of cellulat automata (CA) and their applications in physics, chemistry, biology, and computer science have been reviewed.[1,2] Present very large scale integration (VLSI) levels allow the implementation of elementary CA structures of high complexity, which provide a potential alternative to conventional radix arithmetic processors with highly parallel computation capabililties.[3-5]

Some of the algebraic properties of one-dimensional (1D) CA with periodic boundary conditions have recently been studied by Martin *et al.*,[6] using properties of the CA global state polynomials over finite fields.

Exact results for deterministic CA with periodic boundary conditions

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have also recently been studied by Guan and He,[7] exploiting the properties of circulant matrices on finite fields.

Group and semigroup properties of 1D finite CA, with local rules 90 and 150 (in Wolfram's notation[8]) and null boundary conditions were obtained by Thanailakis *et al.*[9] They observed that the global symmetries of CA with different lengths appear to result in a multiplicity of relationships between the group or semigroup orders and the CA length. However, such relationships should be formally derived, and the present paper contributes in this particular direction.

More specifically, this paper presents a formal treatment of some of the algebraic properties of rule 150, 1D, null-bounded CA. The method is based on the characteristic polynomials of the CA global rule transition matrices, because the operation of the rule 150, as of any other linear (additive) rule, can exactly be described by a matrix. This method, which can very easily be extended to the case of CA with periodic boundary conditions, has been proved very efficient in obtaining algebraic properties of CA as a function of CA length. The main results of the paper are summarized as follows: (i) the general form of the characteristic polynomial of the global rule transition matrix is formally obtained, (ii) mathematical relationships between the CA lengths and the order of the corresponding group-algebraic structures are rigorously determined, and (iii) the semigroup properties are formally derived in the form of recursive relations between the CA lengths and corresponding relations between the orders of the CA semigroup-algebraic structures.

2. DEFINITION OF 1D CELLULAR AUTOMATA

A 1D CA is defined as a uniform linear array of identical cells (sites) of infinite or finite extent with a discrete variable at each site. The global state is completely specified by the values of the variables at each site. In this paper, we exclusively consider finite 1D CA with cell values (local states) $\in \mathbb{Z}_2$ (i.e., 1 or 0), and with no memory associated with the cell beyond the previous time step (clock cycle). The total number of possible global states for a 1D CA of length N is 2^N. A CA evolves in discrete time steps, and the value taken by a particular cell at any given time step is affected by the values of cells in its neighborhood on the previous time step. The neighborhood of a cell is taken, in this paper, to be the cell itself and the cells immediately adjacent to it on the left and right.

Null boundary conditions have been chosen because in VLSI implementations one prefers to hold the end inputs at a constant value, grounded in this particular case.

The local rule may be considered as a Boolean function of the sites within the neighborhood, and may be expressed as

$$T(x) = \beta_{i-1}x^{i-1} * \beta_i x^i * \beta_{i+1}x^{i+1} \tag{2.1}$$

where the symbol $*$ is used to define any binary operation, and the coefficients $\beta \in \{0, 1\}$. The results in this paper were obtained using the following specific form of Eq. (2.1):

$$T(x) = x^{i-1} + x^i + x^{i+1} \quad \text{mod } 2 \tag{2.2}$$

Equation (2.2) defines rule 150 in Wolfram's notation.[8] According to this rule, the value $\alpha_i^{(t)}$ of the cell site i on clock cycle t (CA local state) is given by the relation

$$a_i^{(t)} = a_{i-1}^{(t-1)} + a_i^{(t-1)} + a_{i+1}^{(t-1)} \quad \text{mod } 2 \tag{2.3}$$

3. FORMAL ANALYSIS OF 1D CA

3.1. Representation of the Global Rule

The global state transformation, under the action of the rule 150, in one time step may be represented by the following matrix operation:

$$S^{(t)} = M_N S^{(t-1)} \tag{3.1}$$

where M_N is an $N \times N$ square matrix representing the CA (of length N) global rule for its time evolution, $S^{(t-1)}$ is an $(N \times 1)$ column vector representing the CA global state on clock cycle $(t-1)$, and $S^{(t)}$ is the corresponding column vector representing the CA global state on clock cycle t. The elements of these vectors represent the values of the corresponding cell sites (local states). Note that in the result (3.1), and all subsequent expressions in this paper, mod 2 arithmetic is implied. The global rule transition matrix M_N takes the following form:

$$M_N = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdot & \bigcirc & \\ 0 & 1 & 1 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 0 & 0 & 1 & 1 & 1 & 0 \\ & \bigcirc & \cdot & 0 & 0 & 0 & 1 & 1 & 1 \\ \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}_{N \times N} \tag{3.2}$$

The global state $S^{(t=k)}$ may be obtained from the initial global state $S^{(t=0)}$ using the relation

$$S^{(t=k)} = M_N^k \cdot S^{(t=0)} \tag{3.3}$$

where M_N^k is the kth power of the (mod 2) $N \times N$ rule matrix M_N.

The set of $N \times N$ matrices

$$F = \{M_N, M_N^2, M_N^3, M_N^4,..., M_N^q,..., M_N^k\} \tag{3.4}$$

where k is a positive integer with a value depending on the CA length N, characterizes completely the properties of 1D, null-bounded, CA of different lengths.

It has been proved[9] that the behavior of a rule 150 CA depends on its length N as follows:

(i) If N mod $3 \neq 2$, the set F in Eq. (3.4) forms a cyclic group structure. The corresponding group order is ${}^0G_N = k$, where k is the smallest integer that satisfies the relation

$$M_N^k = I \tag{3.5}$$

and I is the $N \times N$ identity matrix.

(ii) If N mod $3 = 2$, the set F forms a semigroup-algebraic structure of order ${}^0S_N = k - 1$, where k is the smallest integer that satisfies the relation

$$M_N^k = M_N^{q \leqslant k-1} \tag{3.6}$$

Figures 1a and 1b show the types of state transition graphs obtained for rule 150, group- and semigroup-algebraic structures, respectively. The nodes represent the corresponding global states, whereas each arc represents the global rule transition matrix. The action of the rule group symmetry operators (powers of M_N) on a nonsymmetrical initial global state containing a single "1" leads to a cycle of maximum length.[6] Since this mapping is an isomorphism, the number of states in such a cycle is equal to the order ${}^0G_N$ of the corresponding group-algebraic structure. If this action, however, is on a symmetrical initial global state, it leads to a cycle whose length is a divisor of ${}^0G_N$, corresponding to a subgroup order. The value of the parameter q in Eq. (3.6) is equal to the tail-tree height of the corresponding semigroup state transition graph. The value of the parameter k in Eq. (3.6) is equal to the total number of states in the loop (or loops) with the maximum length plus the number of states in each of the tail-trees rooted at the loop states. Finally, the value of $(k - q)$ is equal

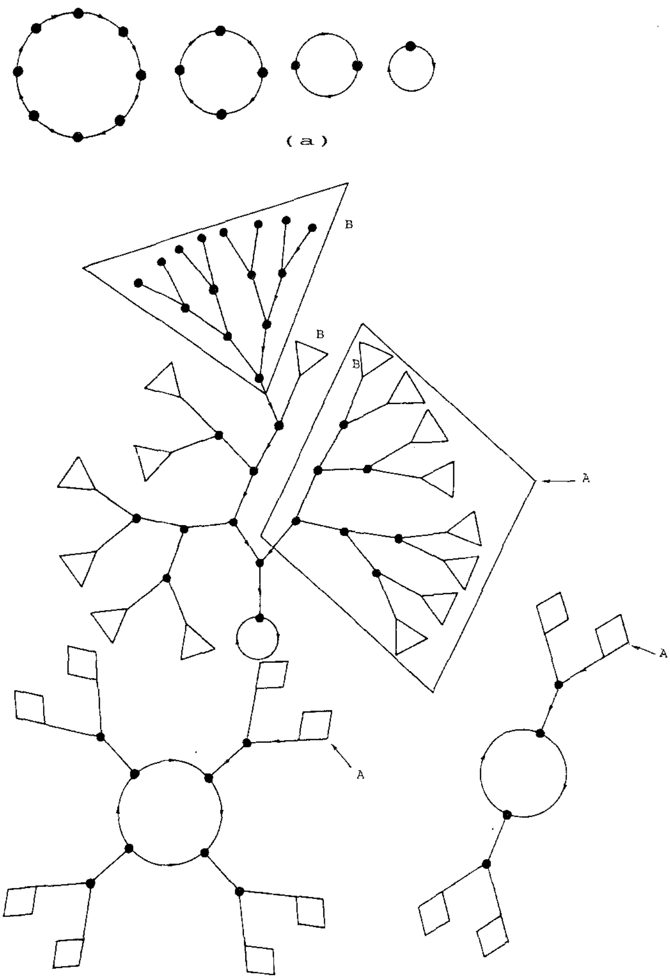

Fig. 1. (a) The global state transition graph for a rule 150, null-bounded, 1D CA of length $N = 7$. Group-algebraic structure of order ${}^0G_N = 8$ (the entire state transition graph consists of 14 cycles of 8 states, 3 cycles of 4 states, 1 cycle of 2 states, and 2 cycles of 1 state). (b) The global state transition graph for a rule 150, null-bounded, 1D CA of length $N = 11$. Semi-group-algebraic structure of order ${}^0S_N = k - 1 = 11$, tail–tree height $q = 8$, and maximum number of loop states $(k - q) = 4$ (the entire state transition graph consists of 1 loop of 4 states, 1 loop of 2 states, and 2 loops of 1 state. At each loop state is rooted a binary tail–tree of height $q = 8$). All triangles are identical and contain the nodes and arcs included in the big triangle named B. Also, all parallelograms are identical (in content) with the big parallelogram named A.

to the number of states in the loop (or loops) with the maximum length, and the order ${}^0S_N$ of the corresponding semigroup-algebraic structure for a CA of length N is equal to the value of the quantity $(k-1)$. It has been observed that the tail–tree lengths are equal for all loops. This property has been already proved for periodic boundary conditions in refs. 6 and 7.

The CA group and semigroup properties may also be studied by employing the characteristic polynomial of the global rule transition matrix, which we will now derive.

Lemma 1. Given a global rule transition matrix M_N for a finite 1D, null-bounded, rule 150 CA of length N, the corresponding characteristic polynomial $P_N(\lambda)$ is given recursively by

$$P_N(\lambda) = (\lambda + 1)\, P_{N-1}(\lambda) + P_{N-2}(\lambda) \tag{3.7}$$

Proof. This result follows directly from the definition of the characteristic polynomial of the global rule matrix M_N.

It is important, however, to be able to obtain directly the nonrecursive form of the characteristic polynomial $P_N(\lambda)$. In this respect, the following theorem holds:

Theorem 1. Given a global rule transition matrix M_N for a null-bounded rule 150 CA of length N, the corresponding characteristic polynomial $P_N(\lambda)$ is given directly by the relation

$$P_N(\lambda) = \frac{1}{2^N} \sum_{j=0}^{j=[N/2]} \binom{N+1}{2j+1} (\lambda+1)^{N-2j} [(\lambda+1)^2 + 4]^j$$

where $[N/2]$ represents the integral part of the number $N/2$, and $\binom{N+1}{2j+1}$ represents the number of all possible combinations of $N+1$ elements into a sequence of $2j+1$ elements.

Proof. Equation (3.7) of Lemma 1 may be written in the general form

$$P_N - aP_{N-1} - P_{N-2} = 0 \tag{3.8}$$

The solution of this finite difference equation is

$$A\rho_1^N + B\rho_2^N = P_N \tag{3.9}$$

with initial conditions

$$P_0 = 1 \qquad \text{and} \qquad P_1 = a \tag{3.10}$$

where ρ_1 and ρ_2 are the roots of the quadratic equation $y^2 - ay - 1 = 0$. From Eqs. (3.9) and (3.10) we determine A and B, and finally obtain

$$P_N = \frac{1}{2^N} \sum_{j=0}^{j=[N/2]} \binom{N+1}{2j+1} a^{N-2j} [a^2+4]^j \tag{3.11}$$

Replacing a in Eq. (3.11) by $(\lambda + 1)$, according to Lemma 1, we get the nonrecursive form of the characteristic polynomial.

3.2. Algebraic Properties

Figure 2 shows the order ${}^0G_N$ of group structures, obtained by simulation, as a function of N. It is apparent from Fig. 2 that the values of ${}^0G_N$

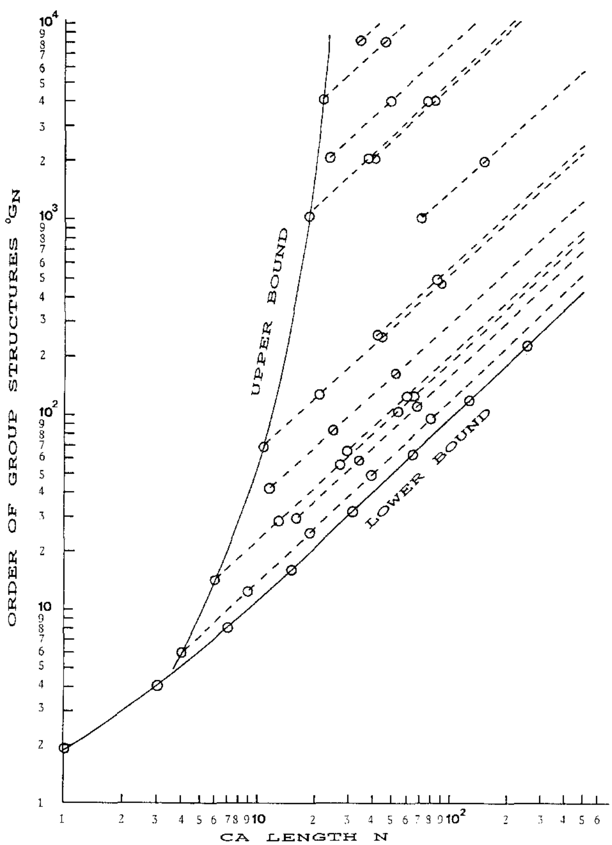

Fig. 2. Orders of group-algebraic structures ${}^0G_N$ for rule 150, null-bounded, 1D CA as a function of the CA length N.

for $N=2^n-1$, where $n=1, 2, 3,...$, constitute the lower bound for the range of allowed values of ${}^0G_N$, whereas the values ${}^0G_N=2(2^{N/2}-1)$ constitute the upper bound. There is a great number of classes of CA satisfying different functional dependences for their lengths and the corresponding group orders. It is highly desirable to have formal results regarding these dependences, and the contribution of the present paper in this respect will now be presented and discussed.

Lemma 2. If $P_N(\lambda)$ is the characteristic polynomial of the global rule transition matrix for a rule 150, null-bounded, 1D CA of length N, then

$$P_N^2(\lambda)=P_N(\lambda^2) \tag{3.12}$$

Proof. The relation is obvious since modulo 2 arithmetic is assumed.

Lemma 3. The characteristic polynomial $P_{2N+1}(\lambda)$, where $N \geqslant 2$, is recursively generated by the relation

$$P_{2N+1}(\lambda)=(\lambda+1)\, P_N^2(\lambda) \tag{3.13}$$

Proof. It can easily be proved by induction on N using Eq. (3.7) of Lemma 1.

Lemma 4. If the length N of a rule 150, null-bounded, 1D CA is even, the corresponding characteristic polynomial $P_N(\lambda)$ has an odd number of terms, whereas if N is odd, $P_N(\lambda)$ has an even number of terms.

Proof. For $N=2$ and $N=3$ the characteristic polynomial $P_N(\lambda)$ takes the form

$$P_2(\lambda)=\lambda^2 \qquad \text{(i.e., an odd number of terms)}$$

and

$$P_3(\lambda)=\lambda^3+\lambda^2+\lambda+1 \qquad \text{(i.e., an even number of terms)}$$

respectively. From Lemma 1 it is obvious that $P_N(\lambda)$ has the same parity as $P_{N-2}(\lambda)$. Hence Lemma 4 follows.

Let us now suppose that the group order of a CA of length $N \bmod 3 \neq 2$ is ${}^0G_N=k$. For this CA we consider a cycle of maximal length obtained from a nonsymmetrical initial global state. The states of such a cycle can be embedded in the states of a CA with length $2N+1$, which have the following bit configurations: the first N cells contain the corresponding global states of the CA with length N; the $(N+1)$ cell contains the bit "0" and the $(N+2)$, $(N+3)$,..., $(2N+1)$ cells contain configurations which are bit reflection-symmetric to those of the above global states considered for the CA of length N. Since the CA global rule is reflection-symmetric, the symmetry of the initial global state is preserved and hence the zero value of the $(N+1)$ cell is also preserved. Therefore, we have a cycle, with symmetric global states only, of length $k = {}^0G_N$ for the CA of length $2N+1$. It is obvious from the additivity property of rule 150[2,6,8] that the length of this cycle is a divisor of that of the maximal cycle of a CA with length $2N+1$, obtained from a nonsymmetrical initial global state containing a single "1," i.e., $k/{}^0G_{2N+1}$. In fact the following theorem holds.

Theorem 2. If ${}^0G_{2N+1}$ and ${}^0G_N$ are the group orders for rule 150, null-bounded, 1D CA of lengths $2N+1$ and N, respectively, where $N \bmod 3 \neq 2$ and ${}^0G_N - 1 \geqslant N$, then

$$ {}^0G_{2N+1} = 2\,{}^0G_N \tag{3.14} $$

Proof. For rule 150 CA group-algebraic structures the characteristic polynomial $P_N(\lambda)$ is

$$ P_N(\lambda) = \lambda^N + c_{N-1}\lambda^{N-1} + \cdots + c_1\lambda + 1 $$

or

$$ \lambda^{k-N} P_N(\lambda) = \lambda^k + c_{N-1}\lambda^{k-1} + \cdots + c_1\lambda^{k-N+1} + \lambda^{k-N} $$

where $k = {}^0G_N$. Since $k-1 \geqslant N$,

$$ \lambda^{k-N} P_N(\lambda) = \lambda^k + P_N(\lambda)\, Q(\lambda) + R(\lambda) \tag{3.15} $$

where $Q(\lambda)$ and $R(\lambda)$ are, respectively, the quotient and remainder of the polynomial division

$$ \frac{P_N(\lambda)\, Q(\lambda) + R(\lambda) \equiv F(\lambda) = c_{N-1}\lambda^{k-1} + \cdots + c_1\lambda^{k-N+1} + \lambda^{k-N}}{P_N(\lambda) = \lambda^N + c_{N-1}\lambda^{N-1} + \cdots + c_1\lambda + 1} \tag{3.16} $$

It can easily be proved that $R(\lambda) = 1$. Therefore, Eq. (3.15) reduces to

$$ \lambda^{k-N} P_N(\lambda) = \lambda^k + P_N(\lambda)\, Q(\lambda) + 1 \tag{3.17} $$

Obviously, the above equation holds only for $k = {}^0G_N$. Since $F(\lambda)$ and $P_N(\lambda)$ always differ by one term, taking into account Lemma 4, simple parity algebra shows that $Q(\lambda)$ contains always an odd number of terms.

Replacing λ in Eq. (3.17) by M^2_{2N+1}, we obtain

$$ M^{2k-2N}_{2N+1} P_N(M^2_{2N+1}) = M^{2k}_{2N+1} + P_N(M^2_{2N+1})\, Q(M^2_{2N+1}) + I \tag{3.18} $$

From Lemma 2, Lemma 3, and the Cayley–Hamilton theorem, we have

$$P_N(M_{2N+1}^2) = M_{2N+1} P_N(M_{2N+1}^2) \tag{3.19}$$

From Eq. (3.18) after the repeated use of Eq. (3.19) and because of the odd parity of $Q(\lambda)$, we obtain

$$M_{2N+1}^{2k} = I \tag{3.20}$$

Because k for the CA of length N is minimum and, also, $k/{}^0G_{2N+1}$, the parameter $2k$ in Eq. (3.20) for the CA of length $2N+1$ is also minimum. Hence,

$${}^0G_{2N+1} = 2k = 2\ {}^0G_N \tag{3.21}$$

It is obvious from Theorem 2 that all even CA lengths $N_{\text{initial}} = N_{m=1}$, such that $N_{\text{initial}} > 1$ and $N_{\text{initial}} \bmod 3 \neq 2$, serve as initial values for corresponding classes, within each of which the following recursive relations hold:

$$N_{m+1} = 2N_m + 1$$

and (3.22)

$${}^0G_{N_{m+1}} = 2\ {}^0G_{N_m}$$

where $m = 1, 2, 3, \ldots$. The evennes condition is merely an efficient way of identifying initial values $N_{\text{initial}} = N_{m=1}$ for corresponding classes of CA satisfying Eqs. (3.22).

Now we will prove the results related to the lower bound described above and shown in Fig. 2.

Lemma 5. The characteristic polynomial $P_N(\lambda)$ of rule 150, null-bounded, 1D CA of length $N = 2^n - 1$, where $n = 1, 2, 3, \ldots$, is given by

$$P_{N=2^n-1}(\lambda) = (\lambda + 1)^N \tag{3.23}$$

Proof. It can easily be proved by induction on n using Eq. (3.13) of Lemma 3.

Theorem 3. If the length of a rule 150, null-bounded, 1D CA is of the form $N = 2^n - 1$, where $n = 2, 3, 4, \ldots$, then the corresponding group order is given by the relation

$${}^0G_{N=2^n-1} = 2^n = N + 1 \tag{3.24}$$

Proof. According to Lemma 5, all coefficients c_r of the characteristic polynomial $P_N(\lambda)$ for $N = 2^n - 1$ are equal to 1.

From Eq. (3.23) of Lemma 5 and the Cayley–Hamilton theorem[10] we have

$$M_N^{2^n-1} = M_N^{2^n-2} + M_N^{2^n-3} + \cdots + M_N + I$$

Hence

$$M_N^{2^n} = I$$

Therefore, the corresponding group order is

$${}^0G_{N=2^n-1} = 2^n = N+1$$

It is important to note that Theorem 3 relates ${}^0G_N$ directly to N, whereas Theorem 2, which also applies to the lower bound of Fig. 2, relates recursively the values of N and the values of ${}^0G_N$. Of course, Theorem 3 comes immediately as a consequence of Theorem 2 with $N_{m=1} = 3 = 2^2 - 1$.

We will now present the results related to semigroup-algebraic structures.

Lemma 6. Given a global rule transition matrix M_N for a rule 150, null-bounded, 1D CA, the corresponding characteristic polynomial $P_N(\lambda)$ is recursively given by

$$P_N(\lambda) = \lambda^4 P_{N-4}(\lambda) + (\lambda+1)^2 P_{N-6}(\lambda)$$

Proof. It can easily be proved by successive application of Lemma 1.

All even CA lengths $N_{\text{initial}} = N_{m=1}$, such that $N_{m=1} \bmod 3 = 2$, serve as initial values for corresponding classes of semigroup-algebraic structures, within each of which the CA lengths satisfy the relation

$$N_{m+1} = 2N_m + 1 \tag{3.25}$$

where $m = 1, 2, 3, \ldots$.

We will now present some results related to such classes of CA semigroup structures.

Theorem 4. If $M_{N_m}^k = M_{N_m}^q$ for a rule 150, null-bounded, 1D CA of length $N_m \bmod 3 = 2$, then for a CA of length $N_{m+1} = 2N_m + 1$ the following relation holds:

$$M_{N_{m+1}}^{2k} = M_{N_{m+1}}^{2q}$$

Proof. The proof is exactly the same as that of Theorem 2, with two exceptions: the characteristic polynomial $P_N(\lambda)$ does not have a constant term and the remainder $\mathrm{R}(\lambda)$ is a monomial of the form $R(\lambda) = \lambda^q$.

It is obvious from Theorem 4 that all lengths such that $N_{m=1} \bmod 3 = 2$ and at the same time $N_{m=1} \bmod 2 = 0$ serve as initial values for corresponding classes, within each of which the following recursive relations hold:

$$N_{m+1} = 2N_m + 1 \tag{3.26}$$

$$q_{N_{m+1}} = 2q_{N_m} \tag{3.27}$$

$$k_{N_{m+1}} = 2k_{N_m} \tag{3.28}$$

$$(k-q)_{N_{m+1}} = 2(k-q)_{N_m} \tag{3.29}$$

$$^0S_{N_{m+1}} = 2\,{}^0S_{N_m} + 1 \tag{3.30}$$

where $m = 1, 2, 3, \ldots$. The initial lengths $N_{\text{initial}} = N_{m=1}$ of the above CA classes are given by

$$\mathrm{N}_{\text{initial}} \equiv N_{m=1} = 2, 8, 14, 20, \ldots, r, r+6, \ldots$$

Theorem 4 and Eqs. (3.28)–(3.30) are not valid for the special case of the pair $N_{m=1} = 2$ and $N_{m+1} = 2N_m + 1 = 5$. This exception is due to the fact that their global rule transition matrices do not satisfy the basic hypothesis of Theorem 4.

Theorem 5. For rule 150, null-bounded, 1D CA semigroup structures with lengths $N_{m=1}$, where $N_{m=1} \bmod 2 = 0$ and at the same time $N_{m=1} \bmod 3 = 2$, the corresponding tail–tree height $q_{N_{m=1}}$ in the state transition graph is

$$q_{N_{m=1}} = 2$$

Proof. The rule 150 CA with length $N \bmod 3 = 2$ defines a semigroup-algebraic structure[9] and thus its characteristic polynomial $P_N(\lambda)$ does not have a constant term, i.e.,

$$P_N(\lambda) = \lambda^N + c_{N-1}\lambda^{N-1} + \cdots c_{q-1}\lambda^{q-1} + \lambda^q \tag{3.31}$$

where $1 \leqslant q \leqslant N$. The values of $N_{m=1}$ that satisfy the relations $N_{m=1} \bmod 2 = 0$ and $N_{m=1} \bmod 3 = 2$ are $N_{m=1} = 2, 8, 14, \ldots, r, r+6, \ldots$.

Theorem 5 follows from Lemma 6 [taking into account that $P_2(\lambda) = \lambda^2$] and Eq. (3.31).

4. CONCLUDING REMARKS

In this paper a formal treatment of some of the properties of deterministic rule 150, null-bounded, 1D CA, for which matrix techniques are applicable, is presented and discussed.

The method used in this paper is based on the characteristic polynomials of the CA global rule transition matrices, which for null-bounded CA are not circulant matrices. This method can very easily be extended to the case of CA with periodic boundary conditions.

The general form of the characteristic polynomial of the global rule transition matrix is derived.

All values of the CA length N correspond to classes of group- or semigroup-algebraic structures with starting values of N which are even integers congruent either to $\{0, 1\}$ mod 3 or 2 mod 3. Within each of these classes the CA lengths satisfy the recursive relation

$$N_{m+1} = 2N_m + 1$$

When the starting value of N is congruent to $\{0, 1\}$ mod 3, the CA belonging to such a class define group-algebraic structures of orders ${}^0G_N$ satisfying the relation

$${}^0G_{N_{m+1}} = 2\,{}^0G_{N_m}$$

whereas when the starting value of N is congruent to 2 mod 3, the CA belonging to such a class define semigroup-algebraic structures described by the relations

$$\begin{aligned} q_{N_{m=1}} &= 2 \\ q_{N_{m+1}} &= 2q_{N_m} \\ k_{N_{m+1}} &= 2k_{N_m} \\ (k-q)_{N_{m+1}} &= 2(k-q)_{N_m} \\ {}^0S_{N_{m+1}} &= 2\,{}^0S_{N_m} + 1 \end{aligned}$$

where q is the tail–tree height of the corresponding semigroup state transition graph, k is the total number of states in the loop (or loops) with the maximum length plus the number of states in each of the tail–trees rooted at the loop states, $(k-q)$ is the number of states in the loop (or loops) with the maximum length, and ${}^0S_N$ is the order of the corresponding semigroup-algebraic structure, being equal to the value of the quantity $(k-1)$.

All allowed values of group orders, ${}^0G_N$, of the corresponding CA algebraic structures lie between a lower bound, defined by ${}^0G_N = N+1$, where $N = 2^n - 1$ and $n = 1, 2, 3,\ldots$, and an upper bound, defined by ${}^0G_N = 2(2^{N/2} - 1)$.

Preliminary work (now in progress) on higher-dimension CA shows that analogous results may also be obtained for such CA.

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